The Modified Crystalline Stefan Problem : Evolution of Broken Facets (Free Boundary Problems)

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The Modified Crystalline Stefan Problem: Evolution of Broken Facets

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1. Introduction

We study a planar model of crystal evolution. This model was derived by M.Gurtin and J.Matias (see [GM]). Its special feature is that the interfacial curve is a polygon. Our aim is to investigate the system when a facet is annihilated or a facet is broken, i.e. a new segment of zero length is inserted.

The system in question was derived from the principles of thermodynamics and after some simplifications takes the form:

$$u_t = \Delta u \quad \text{in} \quad \bigcup_{0<t<T} \Omega_1 \cup \Omega_2,, \tag{1.1}$$

$$[\![\nabla u]\!]\nu_j = -V_j, \quad j = 1, \ldots, N, \quad [\![u]\!] = 0, \tag{1.2}$$

$$\int_{s_j(t)} u = \Gamma_j - \beta_j L_j V_j, \quad j = 1, \ldots, N. \tag{1.3}$$

(1.1) is the heat equation; (1.2) is the law governing the motion of the interface and (1.3) is the counterpart of the Gibbs-Thompson law suitable for polygons in the plane which is supplemented by kinetic undercooling. Originally (1.3) was derived from the balance of capillary forces.

We augment the above system with the initial and boundary conditions

$$s(0) = s_0, \qquad u(0, x) = u_0(x) \tag{1.4}$$

and

$$u|_{\partial\Omega} = 0 \text{ for } t \geq 0. \tag{1.5}$$

We remark here that the above problem was formulated by Herring in the metallurgical literature in the fifties, see [Hr]. Later, it was independently rediscovered by Ben Amar – Pomeau [BP] and Gurtin – Matias [GM]

We remark that system (1.1)-(1.5) has been already studied for regular data, i.e. when all facets have positive length. In this case we have already shown in [Ry1, Ry2] that weak solutions to (1.1)-(1.5) exist and they are unique. We have also established in [Ry2] some geometric properties of the small interfaces.

Let us mention that the above problem for smooth interfaces is also well-posed. This was established in the early '90's, see [CR] and [Ra]. It turns out that $\beta > 0$ is quite important. The problem for $\beta = 0$ and smooth interfaces was studied by Luckhaus [L] and in greater generality by Almgren-Wang [AW]. In particular, they showed that uniqueness fails. Uniqueness is an open problem also if we admit general interfaces for $\beta > 0$, see Soner [S].

Let us also mention that the limiting case of our problem for $u = 0$ is the (driven) 'motion by weighted crystalline curvature'

$$F = \Gamma_j - \beta_j L_j V_j, \quad j = 1, \dots, N \tag{1.6}$$

When the driving term F is zero then above system has been proposed independently by J.Taylor [T1] and S.Angenent–M.Gurtin [AG]. Since then this problem has attracted many authors.

In this paper we deal mainly with the question of existence of solutions (1.1)-(1.5). We show existence of weak solutions in case of initial polygon possessing a number of zero length facets. In order to obtain a tractable problem we consider only zero crystalline curvature facets. On the way we establish existence of solutions up to annihilation of a zero crystalline curvature facets.

We do not show their uniqueness. One of the problems is that the notion of weak solution does not specify the position of a new zero-length facet.

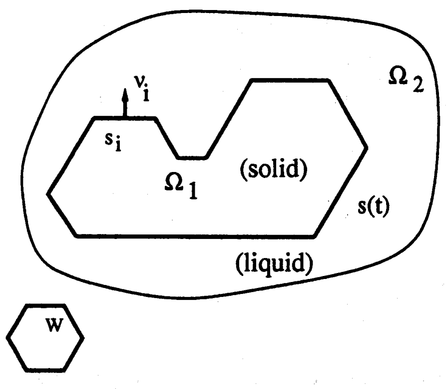
The above results permit a continuation of solutions after each loss of facet and possible creation of new ones. However, this is not quite a corollary implying global existence since we do not consider here all possible topological catastrophes.

On the other hand, evolution past singularity has been already established for motion of polygons by crystalline curvature. The authors of [EGS] and [FG] treated the case of graphs, while in [IS] the case of any closed polygon is covered.

We shall announce a number of results here and we will present the main ideas while asking the Reader willing to learn the full account to refer to the original paper [Ry3].

2. Preliminaries

In order to make the presentation of notation easier we refer to the picture below:



A region with smooth boundary Ω (a vessel) is a sum of $\Omega_1(t)$ (an ice crystal) bounded by interface $s(t)$ and $\Omega_2(t)$ (water), i.e. and $s(t) = \partial\Omega_1(t) \cap \partial\Omega_2(t)$. The facets of s are denoted by s_i, $i = 1, \ldots, N$. L_i is the length of facet s_i, $L = \sum_{j=1}^{N} L_j$; ν is the outer normal to $s(t)$. The jump of ϕ across s is denoted by $[\![\phi]\!]$, i.e.

$$[\![\phi]\!](x_0) = \lim_{\Omega_2(t) \ni x \to x_0} \phi(x) - \lim_{\Omega_1(t) \ni x \to x_0} \phi(x), \quad x_0 \in s(t) = \partial\Omega_1(t) \cap \partial\Omega_2(t).$$

We assume that $\beta_i > 0$ and Γ_i are constants. We shall see momentarily how Γ_i are related to curvature. Before that, we note that in order to fully describe the position of $s(t)$ at time t, it is sufficient to specify the signed distance between the line containing the i–th facet at the initial time and the line containing the i–th facet at the time instant t. We shall denote this distance by $z_i(t)$, hence

$$V_i = \frac{dz_i}{dt}$$

is the facet velocity in the direction of the outer normal. Thus, the vector $\mathbf{z} = (z_1, \ldots, z_N)$ fully describes evolution of the interface.

In our considerations the Wulff shape W plays a role of a reference polygon. We note that if $s(t)$ is convex, then $s(t)$ and W have the same number of facets. We always assume that $s(t)$ is an *admissible*, i.e.

(i) the normals ν_i belong to the set of normals to the Wulff shape W

(ii) normals ν_i to the neighboring facets s_i are normals to the neighboring facets of W.

The constants Γ_i are defined as follows (see [G, §12.5]):

$$\Gamma_i = \begin{cases} -\ell_i & \text{if } s \text{ is locally convex near both ends of } s_i; \\ \ell_i & \text{if } s \text{ is locally concave near both ends of } s_i; \\ 0 & \text{otherwise;} \end{cases}$$

where ℓ_i is the length of the facet of the Wulff shape to which ν_i is outer normal.

Interestingly, Γ_j/L_j is the crystalline weighted curvature of s_j. The relevant definition, which does not need any differential structure of s is given in [T2] p. 423. If z_i are as above and $\mathbf{z} = (z_1, \ldots, z_N)$, i.e. $s(\mathbf{z})$ is a polygon resulting from s by moving entire facet s_i by z_i in the direction of the normal ν_i, then we shall denote by $A(\mathbf{z})$ is the area surrounded by $s(\mathbf{z})$ and by $E(\mathbf{z})$ the surface energy of $s(\mathbf{z})$, i.e. $E(z) = \sum_{i=1}^{N} f(\nu_i) L_i$, where $f(\nu)$ is the surface energy density. Using this notation, the *crystalline weighted curvature* $\mathcal{K}_i$ *of* s_i is

$$\mathcal{K}_i = - \lim_{\Delta z_i \to 0} \frac{E(\mathbf{z} + \mathbf{e}_i \Delta z_i) - E(\mathbf{z})}{A(\mathbf{z} + \mathbf{e}_i \Delta z_i) - A(\mathbf{z})}$$

where $\mathbf{e}_i$, $i = 1, \ldots, N$, are the standard unit vectors of the coordinate axis in $\mathbb{R}^N$. It is not difficult to check that $\mathcal{K}_j = \Gamma_j / L_j$.

In our presentation we shall assume that N is constant in time, but some of the facets may be of zero length.

3. Existence and uniqueness of solutions with fixed number of facets

We have to formulate (1.1)-(1.5) in a weak form. We look for a position $\mathbf{z}$ of the interface as well as for the distribution of temperature u. We impose some minimal regularity, namely we need

1) $\mathbf{z} \in C^1([0,T)\mathbb{R}^N)$;

and

2) $u \in C^\alpha([0,T), H_0^1(\Omega))$, $(1 > \alpha > 0)$, $u_t \in L^\infty_{loc}([0,T), H^{-1}(\Omega))$, such that $u(0) = u_0 \in H_0^1(\Omega)$ and u satisfies the weak form of (1.1)-(1.2):

$$\langle u_t, h\rangle = \int_\Omega \nabla u(x)\cdot\nabla h(x)dx + \sum_{j=1}^N \int_{s_j} V_j h(x)dl, \tag{3.1}$$

for all $h \in H_0^1(\Omega)$, here $\langle\cdot,\cdot\rangle$ is the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and $\frac{dz_j}{dt} = V_j$ fulfill (1.3).

Apparently, the three terms in (3.1) seem loosely related to each other. We shall show that we can simplify their form. We start with the observation that the functionals

$$H_0^1(\Omega) \ni h \mapsto \int_{s_i(t)} h\,dl \in \mathbb{R}$$

are continuous. Therefore, by the Riesz Representation Theorem there is just one element $f_i \in H_0^1(\Omega)$ such that,

$$\int_\Omega \nabla h(x)\cdot\nabla f_i(x)\,dx =: (h, f_i)_{H_0^1(\Omega)} = \int_{s_i(t)} h\,dl. \tag{3.2}$$

It is apparent from (3.2) that in fact, f_i satisfies the equation

$$-\Delta f_i = \delta_{s_i}$$

and hence it belongs to $H_0^1(\Omega)\cap H^\sigma(\Omega)$, $\sigma < 3/2$ (cf. [Ry1])

Thus, the weak form becomes

$$\langle u_t, h\rangle = -(u,h)_{H_0^1(\Omega)} + \sum_{j=1}^N V_j (f_j, h)_{H_0^1(\Omega)}, \quad u(0) = u_0, \tag{3.3a}$$

$$\frac{dz_i}{dt} = \frac{\Gamma_i - (u, f_i)_{H_0^1(\Omega)}}{\beta_i L_i}, \quad z_i(0) = 0, \quad i = 1,\ldots,N. \tag{3.3b}$$

Let us also mention that we need more regular initial data than just belonging to $H_0^1(\Omega)$. We shall say that u_0 is admissible if

$$u_0 - \sum_{i=1}^N V_i(0) f_i(0) \in H^\sigma(\Omega) \quad \text{for all } \sigma < 2$$

In order to be precise we need a slightly different function space which we shall define momentarily.

Lemma 1. *Let us suppose that $\partial\Omega$ is smooth, then the operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ given by $\mathcal{A}u = -\Delta u$ for $u \in D(\mathcal{A}) = H_0^1(\Omega) \cap H^2(\Omega)$ is sectorial and the fractional powers*

$$X^\alpha := (L^2(\Omega))^\alpha$$

are well defined. In particular $X^{1/2} = H_0^1(\Omega)$. Moreover, $\mathcal{A}$ is self-adjoint and strictly positive.

It is an abuse of notation, but we shall write $H^{2\sigma}(\Omega)$ for X^σ.

Now, the existence result may be formulated in a following way:

Theorem 1. [cf. [Ry1-Ry2] *Let us suppose that s_0 is an admissible polygon, u_0 is an admissible data. Then, there exist $T > 0$ and a unique weak solution to (3.3) satisfying*

$$\mathbf{z} \in C^{1,\alpha}([0,T), \mathbb{R}^N), \qquad u \in C^\gamma([0,T), H^\sigma(\Omega)),$$

where $0 < \alpha < \frac{1}{2}$, and $0 < \gamma$, $1 \leq \sigma$ are such that $\gamma + \sigma < 3/2$.

The proof we shall present is new. It contains ideas which will be subsequently used to show existence in the case of broken facets.

We transform system (3.3) by introducing a new variable U:

$$U := -\Delta^{-1} u.$$

Hence, the weak form becomes

$$(U_t, h)_{H_0^1(\Omega)} = (\Delta U, h)_{H_0^1(\Omega)} + \sum_{j=1}^N V_i (f_i, h)_{H_0^1(\Omega)}, \ \forall h \in H_0^1(\Omega)$$

or

$$U_t = \Delta U + \sum_{i=1}^N V_i f_i, \quad U(0) = -\Delta^{-1} u_0, \tag{3.4a}$$

and it is coupled to

$$\frac{dz_i}{dt} = \frac{\Gamma_i + (\Delta U, f_i)_{H_0^1(\Omega)}}{\beta_i L_i}, \quad z_i(0) = 0, \quad i = 1, \ldots, N. \tag{3.4b}$$

Let us notice that system (3.4) is slightly awkward since this is an ODE coupled to heat equation with the highest order term in the ODE. Thus, some additional work is required.

Let us first suppose that the postulated solution exists, i.e. $\mathbf{V} = (V_1, \ldots, V_N) \in C^\alpha$ for some positive α. We know that the map

$$\mathbb{R}^N \ni \mathbf{z} \mapsto f_i(\mathbf{z}) \in H^\sigma(\Omega), \quad \sigma < 3/2$$

is Hölder continuous with exponent α satisfying $\alpha+\sigma<3/2$, (see [Ry1, Proposition 3.4]). Thus, $t \mapsto f_i(\mathbf{z}(t))$ is Hölder continuous. Then, by [He, Theorem 3.2.2] the constant variation formula is valid:

$$U(t) = -e^{\Delta t}\Delta^{-1}u_0 + \int_0^t e^{\Delta(t-\tau)} \sum_{i=1}^N f_i(\tau)V_i(\tau)\,d\tau.$$

and due to regularity of f_i's we may recover u:

$$u(t) = e^{\Delta t}u_0 - \int_0^t \Delta e^{\Delta(t-\tau)} \sum_{i=1}^N f_i(\tau)V_i(\tau)\,d\tau. \tag{3.5}$$

Further manipulations require working with the Green function G for the heat operator, i.e. with a function satisfying $(\partial_t - \Delta_x)G(x,y,t) = \delta_y$ and the boundary condition (1.5). It is a well-known fact that $G(x,y,t)$ is equal to the sum of the Gauss-Weierstrass kernel $(4\pi t)\exp(-(x-y)^2/4t)$ and a correcting smooth term $H(x,y,t)$ whose specific form is not quite important to us.

One of the properties of G is the formula (cf. [Ry2])

$$e^{\Delta t}u_0(x) = \int_\Omega G(x,y,t)u_0(y)\,dy.$$

For a special case of u_0 more can be proven, see Lemma 3.5 in [Ry2]:

$$\Delta e^{\Delta t} f_i(x) = -\int_{s_i} G(x,y,t)\,dy, \quad t>0. \tag{3.6}$$

If we insert (3.5) into (3.3b), and we use (3.6), then we come to

$$V_i = \frac{\Gamma_i}{\beta_i L_i} - \frac{1}{\beta_i L_i}\int_{s_i(t)}\int_\Omega G(x,y,t)u_0(y)\,dy\,dx + \frac{1}{\beta_i L_i}\sum_{j=1}^N \int_0^t M_{ij}(\mathbf{z}(t),\mathbf{z}(\tau),(t-\tau))V_j\,d\tau, \tag{3.7}$$

where

$$M_{ij}(\mathbf{z}_1,\mathbf{z}_2,\zeta) = \int_{s_i(\mathbf{z}_1)}\int_{s_j(\mathbf{z}_2)} G(x,y,\zeta)\,dxdy. \tag{3.8}$$

Thus, we have obtained an integral equation for $\mathbf{V}$. We can denote the RHS of (3.7) by $\Psi(\mathbf{V})$ and note that Ψ is a continuous operator on $C([0,T];\mathbb{R}^N)$ into itself. Thus, the problem of finding a solution to (3.3) is reduced to a fixed point problem: $\mathbf{V}=\Psi(\mathbf{V})$. In order to proceed we need an estimate for M_{ij}. Namely, we proved in [Ry2, Lemma 3.7] that for $\zeta>0$ we have

$$|M_{ij}(\mathbf{z}_1,\mathbf{z}_2,\zeta) - M_{ij}(\mathbf{z}_1',\mathbf{z}_2',\zeta)| \le \frac{C}{\zeta^{1/2}}(|\mathbf{z}_1-\mathbf{z}_1'| + |\mathbf{z}_2-\mathbf{z}_2'|), \tag{3.9}$$

where C is independent of ζ.

Now, we can apply the Banach Fixed Point Theorem to (3.7) for sufficiently small $T > 0$ to find a solution in the space of continuous functions.

Finally, we have to show that a unique solution to (3.7) is in fact Hölder continuous. We will deal only with the core of the problem, i.e. with the second and third terms of the RHS of (3.7). We adopt the standard notation: $\Delta_h v(t) = v(t+h) - v(t)$. Let us set

$$N_i(t) := \int_{s_i(t)} \int_\Omega G(x,y,t) u_0(y)\, dy\, dx.$$

We note that

$$\begin{aligned}\Delta_h N_i(t) &= \int_{s_i(t+h)} \int_\Omega G(x,y,t+h) u_0(y)\, dy\, dx - \int_{s_i(t)} \int_\Omega G(x,y,t+h) u_0(y)\, dy\, dx \\ &+ \int_{s_i(t)} \int_\Omega (G(x,y,t+h) - G(x,y,t)) u_0(y)\, dy\, dx \\ &= J_1 + J_2.\end{aligned}$$

Basing on the observation that $\int_\Omega G(x,y,t+h) u_0(y)\, dy$ belongs to $H_0^1(\Omega)$ we use [Ry0, equation (2.16)] to deduce that

$$|J_1| \le C|\mathbf{z}(t+h) - \mathbf{z}(t)|^{1/2} \|e^{\Delta t} u_0\|_{H_0^1(\Omega)} \le C h^{1/2} \|u_0\|_{H_0^1(\Omega)}.$$

As far as J_2 is concerned we shall show only the most difficult part related to estimating the singular part of G, i.e. the Gauss-Weierstrass kernel. Let us set

$$J_2' = \frac{1}{4\pi t} \int_{s_i(t)} \int_\Omega e^{\frac{-(x-y)^2}{4(t+h)}} u_0(y)\, dy dx - \frac{1}{4\pi t} \int_{s_i(t)} \int_\Omega e^{\frac{-(x-y)^2}{4t}} u_0(y)\, dy dx$$

We notice that

$$\begin{aligned}J_2' &= \frac{1}{4\pi t} \int_{s_i(t)} \int_\Omega \frac{-h}{t+h} e^{\frac{-(x-y)^2}{4(t+h)}} u_0(y)\, dy dx \\ &- \frac{1}{4\pi t} \int_{s_i(t)} \int_\Omega e^{\frac{-(x-y)^2}{4t}} (1 - \exp(-(x-y)^2 4((t+h)^{-1} - t^{-1}))) \\ &= \frac{1}{4\pi}(J_{21}' + J_{22}')\end{aligned}$$

The maximum principle applied to J_{21}' yields

$$|J_{21}'| \le \frac{h}{t+h} \int_{s_i(t)} \|u_0\|_{L^\infty(\Omega)} \le \frac{h L_i(t)}{t} \|u_0\|_{L^\infty(\Omega)} \le C \|\mathbf{V}\|_{C[0,T]} \|u_0\|_{L^\infty(\Omega)}.$$

For the purpose of estimating J_{22}' we make use of

$$1 - e^{-|x|} \le C_\alpha |x|^\alpha, \quad 0 < \alpha < 1,$$

whose proof we leave to the Reader (cf. also [Ry3, equation (4.16)]). This inequality implies that

$$|J_{22}'| \le \int_{s_i(t)} \int_\Omega \frac{e^{-(x-y)^2}}{4t} u_0(y)(x-y)^{2\alpha} h^\alpha (t+h)^{-\alpha} t^{-\alpha}\, dy dx,$$

where $\alpha \le 1/2$. Now, the same reasoning as applied to J_{21}' gives us the following estimate:

$$\begin{aligned} |J_{22}'| &\le (\operatorname{diam}\Omega)^\alpha h^\alpha (t+h)^{-\alpha} t^{-\alpha} \int_{s_i(t)} \|u_0\|_{L^\infty(\Omega)} \\ &\le C(\operatorname{diam}\Omega)^\alpha h^\alpha \|u_0\|_{L^\infty(\Omega)} \|\mathbf{V}\|_{C[0,T]} t^{1-2\alpha}. \end{aligned}$$

Finally,

$$|N_i(t+h) - N_i(t)| \le Ch^{1/2}.$$

Now, we turn our attention to estimating the third term of (3.7) and let us set

$$\sum_{j=1}^N \int_0^t M_{ij}(\mathbf{z}(t), \mathbf{z}(\tau), t-\tau) V_j(\tau)\, d\tau =: F_i(t).$$

We shall estimate the difference $\Delta_h \mathbf{F} := \sum_{i=1}^N |F_i(t+h) - F_i(t)|$:

$$\begin{aligned} F_i(t+h) - F_i(t) &= \sum_{j=1}^N \int_0^h M_{ij}(\mathbf{z}(t+h), \mathbf{z}(\tau), t+h-\tau) V_j(\tau)\, d\tau \\ &+ \sum_{j=1}^N \int_0^t (M_{ij}(\mathbf{z}(t+h), \mathbf{z}(\tau+h), t-\tau) V_j(\tau+h) - M_{ij}(\mathbf{z}(t), \mathbf{z}(\tau), t-\tau) V_j(\tau))\, d\tau \end{aligned}$$

Hence, by (3.9)

$$\begin{aligned} |F_i(t+h) - F_i(t)| &\le \int_0^h \frac{C}{\sqrt{t+h-\tau}}\, d\tau + \int_0^t C \frac{\max_\tau |V_j(\tau+h) - V_j(\tau)|}{\sqrt{t+h-\tau}} \max_\xi |z(\xi)|\, d\tau \\ &+ \sum_{j=1}^N \int_0^t |\mathbf{z}(\tau+h) - \mathbf{z}(\tau)| \frac{\max_\tau |V_j(\tau+h)|}{\sqrt{t+h-\tau}}\, d\tau \\ &\le C\sqrt{h} + CT^{3/2} \max_j |V_j(\tau)| \max_\tau |V_j(\tau+h) - V_j(\tau)| \\ &+ CT^{3/2} \max_\tau \max_j |V_j(\tau)|^2 h. \end{aligned}$$

We notice the presence on the RHS of a quantity which we are estimating, i.e. $\max_\tau |V_j(\tau+h) - V_j(\tau)|$. We use eq. (3.7) again and we arrive at (after noting that $\mathbf{z} \mapsto L_i$ is Lipschitz continuous):

$$|\Delta_h \mathbf{F}| \le C\sqrt{h} + CT^{3/2} \max_\tau |\Delta_h \mathbf{F}| + C \max_i \|V_i\|^2_{L^\infty([0,T])} T^{3/2} h$$

and for $CT^{3/2} \le 1/2$

$$\|\Delta_h \mathbf{F}\|_{L^\infty([0,T])} \le C\sqrt{h} + C(\max_i \|V_i\|^2_{L^\infty([0,T])} + \frac{1}{2}\|\Delta_h \mathbf{F}\|_{L^\infty([0,T])}$$

that is we obtain

$$\max_\tau |F(\tau + h) - F(\tau)| \le C\sqrt{h}$$

as desired.

Once we have established Hölder continuity of V_i's we can recover u from (3.5).

4. Properties of solutions when facets vanish

We wish to establish time regularity of weak solutions to (3.3) near the instant of collapsing of a facet. Our experience with parabolic problems suggests that space regularity will play an important role. We recall that $f_i \in H^\sigma(\Omega)$, $\sigma < 3/2$ and it is not true that $s = 3/2$. We stress that $V_i \in C^\alpha$, and $u \in C^\alpha([0,T), H^\sigma(\Omega))$, where $\alpha < \frac{1}{2}$ and $\alpha + \sigma < 3/2$.

We shall denote $\mathcal{Z}$ a nonempty subset of $\{1, \dots, N\}$, such that

(a) if $i \in \mathcal{Z}$, then $\Gamma_i = 0$;
(b) its complement $\mathcal{Z}^c = \{1, \dots, N\} \setminus \mathcal{Z}$ is non-void.

Here is the main result of Section 4.

Theorem 2. *Let us suppose that $\mathcal{Z}$ is as above, $(\mathbf{z}, u)$ is a weak solution to (3.3) such that u_0 is admissible and*

$$\lim_{t\to T} L_i(t) = 0, \qquad i \in \mathcal{Z},$$

$$\lim_{t\to T} L_i(t) \ge m > 0, \qquad i \in \mathcal{Z}^c.$$

Then,

(i) $\lim_{t\to T} \mathbf{V}(t)$ *exists;*
(ii) $\lim_{t\to T} u(t)$ *exists in* $H^\sigma(\Omega)$ *for each* $\sigma < 3/2$, *in particular*

$$\sup_{t\in[0,T)} \|u(t)\|_{L^\infty(\Omega)} \le C < \infty.$$

Moreover, if $\mathbf{V} \in C^\alpha([0,T], \mathbb{R}^N)$ *for all* $\alpha < 1/4$, *then for all* $t \in [0,T]$ *we have*

$$u(t) - \sum_{i=1}^N f_i(\mathbf{z}(t))V_i(t) \in H^\sigma(\Omega), \quad \forall \sigma < 2. \tag{4.1}$$

Remark. We stress that we assume Hölder continuity **up to** the instant of vanishing of some facets.

Proof. We shall proceed in a number of steps. We will present the main ideas while referring the Reader to [Ry3] for details.

Step 1. We first establish uniform in t bounds on $\|u\|_{L^\infty(\Omega)}$ and $|\mathbf{V}|$. We use the integral representation of u:

$$u(t) = e^{\Delta t}u_0 - \int_0^t \Delta e^{\Delta(t-\tau)} \sum_{i=1}^N f_i(\tau)V_i(\tau)\, d\tau. \tag{4.2}$$

It is relatively convenient to estimate the $H^\sigma(\Omega)$ norm of the RHS. For this purpose we recall [Ry1, Proposition 3.3]

$$\|f_i(\mathbf{z})\|_{H^\sigma(\Omega)} \le CL_i^\alpha, \quad \alpha+\sigma<3/2 \tag{4.3}$$

and [He, Theorem 1.3.4]

$$\|(-\Delta)^\sigma e^{\Delta t}\|_{L^2(\Omega)} \le Ct^{-\sigma}. \tag{4.4}$$

We also split the set of summation indices in (4.2), subsequently we use (4.3), (4.4) and Lemma 1 to obtain

$$\begin{aligned}\|u(t)\|_\sigma \le Ce^{-\lambda t}\|u_0\|_\sigma + C\int_0^t (t-\tau)^{-1+\delta}e^{-\lambda(t-\tau)}\sum_{i\in\mathcal{Z}} L_i^{\alpha-1}(\tau)\left|\int_{s_i} u\,dl\right| d\tau \\ + C\int_0^t \frac{e^{-\lambda(t-\tau)}}{(t-\tau)^{1-\delta}}\sum_{i\in\mathcal{Z}^c} L_i^{\alpha-1}(\tau)\left(|\Gamma_j| + \left|\int_{s_i} u\,dl\right|\right) d\tau,\end{aligned}$$

where $\sigma/2+\delta<3/4$, and $\delta>0$ is arbitrary. Due to Sobolev embedding,

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_\sigma,$$

for $\sigma>1$ we obtain an estimate for $\|u\|_{L^\infty(\Omega)}(t)$ for which a generalized Gronwall inequality (see [He, Lemma 7.1.1]) is applicable yielding a bound for $\max_{t\in[0,T]}\|u\|_{L^\infty(\Omega)}(t)$. A uniform bound on $|\mathbf{V}|$ may now be easily obtained from (3.3b).
Step 2. We show that the limit

$$\lim_{t\to T}\mathbf{V}(t)$$

exists. For this purpose we use again the constant variation formula (3.7).

$$V_i = \frac{\Gamma_i}{\beta_i L_i} - \frac{1}{\beta_i L_i}\int_{s_i(t)}\int_\Omega G(x,y,t)u_0(y)\,dydx - \frac{1}{\beta_i L_i}\sum_{j=1}^N\int_0^t M_{ij}(\mathbf{z}(t),\mathbf{z}(\tau),t-\tau)V_j\,d\tau, \tag{4.5}$$

The difficulty is the presence of the factor $1/L_i(t)$ which goes to infinity for $i\in\mathcal{Z}$. But the second and third terms of (4.5) are averages and this makes them tractable. Namely, we can represent s_i as follows $s_i=\mu_i[0,L_i]+v_i$, where $v_i,\mu_i\in\mathbb{R}^2$ and μ_i has a unit length, and v_i is a position of one vertex of s_i. Then, the second term takes the form

$$I = \frac{1}{\beta_i}\int_0^1\int_\Omega G(x,\mu_i\cdot\xi L_i(t)+v_i(t),t)u_0(x)\,dxd\xi.$$

It is relatively easy to see that $v_i(t)$ and $L_i(t)$ have a limit as $t\to T$ (see [Ry3] for details). Therefore I has a limit as $t\to T$ because of Lebesgue theorem. The third term can be handled in a similar way (see [Ry3]).

Existence of the limit $\lim_{t\to T}u(t)$ in $H^\sigma(\Omega)$ is easier and left to the Reader (see also

Step 3. In order to estimate

$$\|u(t) - \sum_{j=1}^{N} f_j(\mathbf{z}(t))V_j(t)\|_{H^\sigma(\Omega)}, \quad \sigma < 2$$

we use (3.7). We set up the following identity for an easy application of the essential fact that $V_i \in C^\alpha([0,T])$:

$$\begin{aligned} u(T) &- \sum_{i=1}^{N} V_i(T) f_i(\mathbf{z}(T)) \\ &= e^{\Delta T}(u_0 - \sum_{i=1}^{N} V_i(T) f_i(\mathbf{z}(T))) - \int_0^T \Delta e^{\Delta(T-\tau)} \sum_{i=1}^{N} (V_i(\tau) - V_i(T)) f_i(\mathbf{z}(\tau))\, d\tau \\ &- \int_0^T \Delta e^{\Delta(T-\tau)} \sum_{i=1}^{N} V_i(T)(f_i(\mathbf{z}(\tau)) - f_i(\mathbf{z}(T)))\, d\tau \end{aligned}$$

Now, one can check that all terms belong to $H^\sigma(\Omega)$, for all $\sigma < 2$.

Remark. We have to establish the Hölder continuity of $\mathbf{V}$. This can be achieved with the method of Theorem 1 used for estimating $\Delta_h \mathbf{F}$ but it has to be refined. This will be presented in the next section. Here we note:

Proposition 1. *If at $t = T$ facets with indices i in $\mathcal{Z}$ disappear, then*

$$\mathbf{V} \in C^\alpha([0,T], \mathbb{R}^N), \qquad \forall \alpha < 1/4.$$

5. Evolution of broken facets

The results of previous Section suggest what kind of solution we should look for if we allow facet breaking.

Theorem 3. *Suppose that u_0 is admissible, $\mathcal{Z}$ defined as above is non-empty, s_0 is an admissible polygon, with a number of point (different from vertices) marked and dubbed zero-length zero-curvature facets, their indices form the set $\mathcal{Z}$. Then, there is $T > 0$, such that $\mathbf{z} \in C^{1,\alpha}([0,T); \mathbb{R}^N)$, $u \in C^{0,\alpha}([0,T); H^\sigma(\Omega))$, for all $\alpha \in (0,1/4)$, $\sigma + \alpha < 3/2$ such that $(\mathbf{z}, u)$ is a weak solution to (3.3).*

Let us first remark that for a singular problem we can indeed guarantee lower temporal smoothness of $\mathbf{V}$ than in case of regular data. However, it is not clear if this is a deficiency of the method of the proof or a genuine phenomenon. On the other hand *some* temporal Hölder regularity of $\mathbf{V}$ is necessary for the whole method to work.

Idea of the proof. We shall proceed as in the proof of Theorem 1. Let us suppose that u is a postulated solution. Then, the temporal Hölder continuity implies that (3.7) holds. We shall treat (3.7) as an integral equation for velocities $\mathbf{V}$.

Let us introduce some convenient notation:

$$(\mathcal{D}(\mathbf{V}))_i = \begin{cases} \frac{\Gamma_i}{L_i\beta_i}, & \text{if } i \in \mathcal{Z}^c, \\ 0, & \text{if } i \in \mathcal{Z}; \end{cases}$$

$$(\mathcal{L}(\mathbf{V}))_i(t) = -\frac{1}{\beta_i L_i}\int_{s_i(t)}\int_\Omega G(x,y,t)u_0(y)\,dydx;$$

$$(\mathcal{N}(\mathbf{V}))_i(t) = -\sum_{j=1}^N \int_0^t \int_{s_i(t)}\int_{s_j(\tau)} G(x,y,t-\tau)\frac{V_j(\tau)}{\beta_i L_i}\,dydxd\tau,$$

where $i = 1,\dots,N$.

These definitions are correct even if $L_i(t) = 0$. One can see that

$$\mathcal{D},\ \mathcal{N},\ \mathcal{L} : C([0,T];\mathbb{R}^N) \to C([0,T];\mathbb{R}^N)$$

are continuous and their sum $\mathcal{D}+\mathcal{N}+\mathcal{L}$ is compact. Moreover, one can also check that Schauder fixed point theorem is applicable. Thus, it yields existence of at least one solution to

$$\mathbf{V} = \mathcal{D}(\mathbf{V}) + \mathcal{N}(\mathbf{V}) + \mathcal{L}(\mathbf{V}). \tag{5.1}$$

Subsequently, one may show that all fixed points of $\mathcal{D}+\mathcal{N}+\mathcal{L}$ are Hölder continuous. Thus, solutions to (5.1) (i.e. (3.7)) belong to the class of functions for which this representation was derived.

Let us comment on compacteness of $\mathcal{N}$. We showed (see [Ry2]) that for $i,j \in \mathcal{Z}$, $M_{ij}(\mathbf{z}_1,\mathbf{z}_2,\zeta)$ is locally Lipschitz continuous in $\mathbf{z}_1,\mathbf{z}_2$ with Lipschitz constant bounded by $C\zeta^{-1/2}$ (c.f (3.9)). But this is impossible if i or $j \in \mathcal{Z}$ because of vanishing denominator in $\frac{1}{L_i(t)}$. We set a more restricted goal. We set

$$\bar{M}_{ij}(\mathbf{z}_1,\mathbf{z}_2,\zeta) = \frac{1}{\beta_i L_i(\mathbf{z}_1)}\int_{s_i(\mathbf{z}_1)}\int_{s_j(\mathbf{z}_2)} G(x,y,\zeta)\,dxdy.$$

Our aim is to show that

$$|\bar{M}_{ij}(\mathbf{z}_1,\mathbf{z}_2,\zeta)| \le \frac{C}{\sqrt{\zeta}}(|\mathbf{z}_1| + |\mathbf{z}_2|). \tag{5.2}$$

This is sufficient to establish the compactness of $\mathcal{N}$ due to the general theory, (see [HP, Theorem 7.6.2]).

It is rather clear that we have to deal mainly with the singular part of $G(x,y,t)$ i.e. we study

$$\bar{M}'_{ij}(\mathbf{z}_1,\mathbf{z}_2,\zeta) = \frac{1}{L_i(\mathbf{z}_1)}\int_{s_i(\mathbf{z}_1)}\int_{s_j(\mathbf{z}_2)} K_\zeta(x-y)\,dxdy,$$

where $K_\zeta(z) = (4\pi\zeta)^{-1}\exp(-z^2/4\zeta)$ is the fundamental solution of the heat equation. Once we notice that due to the fixed inner angles of $s(t)$ we can estimate uniformly from below the expression $(x-y)^2$ for x, y belonging to facets of $s(t)$, then the remaining calculations are fairly standard, (see [Ry3] for details).

We establish compactness of $\mathcal{L}(\mathbf{V})$ by establishing its Hölder continuity with exponent $1/2$ and constant independent of $\mathbf{V}$ for $\mathbf{V}$ bounded in $C[0,T]$. Once we have established compactness of the operator on the RHS of (5.1), then in a routine manner using Schauder fixed point we can establish existence of a solution to (5.1), (see [Ry3]).

After we have shown existence of $\mathbf{V}$ a fixed point to (5.1), then the key point is to prove its Hölder continuity. In fact is remains to show that for $\mathcal{N}(\mathbf{V})$ is Hölder continuous with exponent less than $1/4$. The main task is to establish this for $\bar{M}'_{ij}$ defined above. We set up the difference for

$$\int_0^t \sum_{j=1}^N \bar{M}'_{ij}(\mathbf{z}(t), \mathbf{z}(\tau), t-\tau) V_j(\tau)\, d\tau =: J(t).$$

After obvious transformations we obtain:

$$\begin{aligned}
&J(t+h) - J(t)\\
&= \int_t^{t+h} \frac{1}{L_i(t+h)} \int_{s_i(t+h)} \sum_{j=1}^N \int_{s_j(\tau)} \frac{1}{t+h-\tau} \exp\left(-\frac{(x-y)^2}{t+h-\tau}\right) V_j(\tau)\, dx dy d\tau\\
&+ \int_0^t \left(\frac{1}{L_i(t+h)} \int_{s_i(t+h)} - \frac{1}{L_i(t)} \int_{s_i(t)}\right) \sum_{j=1}^N \int_{s_j(\tau)} \frac{V_j(\tau)}{t+h-\tau} \exp\left(-\frac{(x-y)^2}{t+h-\tau}\right) dx dy d\tau\\
&+ \int_0^t \sum_{j=1}^N \frac{1}{L_i(t)} \int_{s_i(t)} \frac{\exp\left(-\frac{(x-y)^2}{t+h-\tau}\right) - \exp\left(-\frac{(x-y)^2}{t-\tau}\right)}{t+h-\tau} V_j(\tau)\, dx dy d\tau\\
&+ \int_0^t \sum_{j=1}^N \frac{1}{L_i(t)} \int_{s_i(t)} \exp\left(-\frac{(x-y)^2}{t-\tau}\right) \left(\frac{1}{t+h-\tau} - \frac{1}{t-\tau}\right) V_j(\tau)\, dx dy d\tau
\end{aligned}$$

The estimates of these four terms are tiresome, we refer the Reader for details to [Ry3].

Our task is finished with showing that solutions to (5.1) yield weak solutions of (3.3), i.e. we have to define u using (4.2).

We note that Schauder fixed point theorem does not guarantee uniqueness of a solution.

6. References

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